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On the Relation Between the Holomorphic Prepotential *and* the Quantum Moduli in SUSY Gauge Theories

J. Sonnenschein*, S. Theisen† and S. Yankielowicz*

* School of Physics and Astronomy
Beverly and Raymond Sackler Faculty of Exact Sciences
Tel–Aviv University
Ramat–Aviv, Tel–Aviv 69978, Israel

†Sektion Physik
Universität München
Theresienstraße 37, 80333 München, FRG

We give a simple proof of the relation $\Lambda\partial_\Lambda\mathcal{F} = \frac{i}{2\pi}b_1\langle\mathrm{Tr}\phi^2\rangle$, which is valid for $N = 2$ supersymmetric QCD with massless quarks. We consider $SU(N_c)$ gauge theories as well as $SO(N_c)$ and $SP(N_c)$. Aa analogous relation which corresponds to massive hypermultiplets is written down. We also discuss the generalizations to $N = 1$ models in the Coulomb phase.

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A lot of activity has followed the beautiful work of Seiberg and Witten [1] on the exact non-perturbative low energy effective action (in the Coulomb phase) of the pure and QCD-like $SU(2)$ $N=2$ supersymmetric gauge theories. In [2] it was generalized to $SU(N_C)$ $N=2$ theories and in [3][4] to $SU(N_C)$ $N=2$ theories with matter in the fundamental representation. Recently this work has been extended to $SO(N_C)$ and $Sp(N_C)$ gauge groups[5][6][7].

In the present letter we prove and discuss relations between the prepotential $\mathcal{F}$ and the quantum moduli of the $N=2$ theory. The most interesting relation reads

$$\Lambda\frac{\partial}{\partial\Lambda}\mathcal{F} = \frac{i}{2\pi}b_1\langle\mathrm{Tr}\phi^2\rangle \tag{1}$$

where ϕ is the adjoint complex scalar in the $N=2$ gauge multiplet, and b_1 is the one-loop coefficient of the beta-function. This relation holds for all $N=2$ theories, either pure or with massless matter quarks. For the case of pure $SU(2)$ this relation is essentially proven in [8] where the modular transformations of the prepotential $\mathcal{F}$ are considered. In [9] the generalization of the Seiberg-Witten approach to $N=2$ string theory is investigated. In particular, the exact non-perturbative result on pure $SU(2)$ and $SU(3)$ $N=2$ Yang-Mills theory were recovered from the tree-level Type II string theory at the corresponding points in moduli space, in the limit of $\alpha' \to 0$, where gravity is decoupled. In this work it was observed that starting from the local case $u \equiv \frac{1}{2}\langle\mathrm{Tr}\phi^2\rangle$ behaves as a period and the relation (1) holds with the dilaton playing the role of Λ. This relation turns out to be crucial in obtaining the rigid theory from the local one.

In the pure $N=2$ gauge theory, the low energy effective action up to terms with two derivatives is completely determined by one holomorphic function of $N=2$ chiral superfields $\mathcal{A}_i$, the prepotential $\mathcal{F}(\mathcal{A})$. For $N_f > 0$, we also have to include (matter) hypermultiplets, whose contribution to the low energy effective action is not determined by a holomorphic structure. However, for the purpose of this note, we won't need their couplings. For the massless case, the perturbative piece of the prepotential is

$$\Big(\frac{(\vec{\alpha}\cdot\vec{\mathcal{A}})^2}{\Lambda^2}\Big)\mathcal{F}_{\text{pert.}}(\mathcal{A}) = \frac{1}{2\pi i}\frac{l(\text{adj.}) - \sum_i l_i(\text{matter})}{l(\text{adj.})}\sum_{\alpha>0}(\vec{\alpha}\cdot\vec{\mathcal{A}})^2\ln\Big(\frac{(\vec{\alpha}\cdot\vec{\mathcal{A}})^2}{\Lambda^2}\Big) \tag{2}$$

The sum is over all positive roots and $l(\text{adj.})$ is the index of the adjoint representation of the gauge group G whereas $l_i(\text{matter})$ is the index of the representation of the i'th matter hypermultiplet. From this expression the perturbative beta-function, which is purely one-loop, follows.

The prepotential may be considered as a holomorphic function of the chiral superfields $\mathcal{A}_i$ *and* the scale Λ. Defining $a_i = \mathcal{A}_i|_{\theta=0}$ and $a_{D_i} = \frac{\partial \mathcal{F}(a)}{\partial a_i}$, one then finds that (a_i, a_{D_i}) are the periods of an abelian differential of the second kind (having poles with zero residue) for the case of $N_f \geq 0$ *massless* hypermultiplets or of the third kind (having poles with non-zero residue) for $N_f > 0$ *massive* hypermultiplets. These differentials are defined on an (auxiliary) hyperelliptic Riemann surface Σ_r of genus $r = \mathrm{rank}(G)$ and the periods are with respect to a symplectic homology basis with one-cycles (α_i, β_i). The Riemann surfaces for pure $SU(N_C)$ [2], $SU(N_C)$ with hypermultiplets [3][4], $SO(N_C)$ without [5] and with [6][7] matter, and finally also for $Sp(N_C)$ [7] have been found by now. In particular ref.[7] gives curves with genus equal to the rank of G. The hypermultiplets were always chosen in the defining representation and their number such that the theory is either asymptotically free or has vanishing beta function. Recently curves for certain $N = 1$ supersymmetric theories were considered in [10][4][11] with matter in the adjoint and/or fundamental representations. We first treat $N = 2$ theories with $G = SU(N_C)$. The remaining classical groups and some $N = 1$ cases will be dealt with below.

The Riemann surface for $SU(N_C)$ is the genus $N_C - 1$ hyperelliptic curve $\Sigma_{N_C - 1}$

$$y^2 = W^2 + F \tag{3}$$

where

$$W = \langle \det(x\mathbf{1} - \phi) \rangle \equiv x^{N_C} - \sum_{k=2}^{N_C} s_k x^{N_C - k} \tag{4}$$

$F = F(x, m_j, \Lambda)$ is a polynomial of its arguments, independent of the s_i and $F(x) \sim x^{N_f}$ for large x. If we parametrize $\langle \phi \rangle = \sum_i a_i H_i$ where H_i are the generators in the Cartan subalgebra, we get in the semiclassical limit $s_2 = \frac{1}{2} a_i a_j \mathrm{Tr}(H_i H_j)$. The exact (non-perturbative) expression is $s_2 = u = \frac{1}{2}\langle \mathrm{Tr}\phi^2 \rangle$ where ϕ is the Higgs field, i.e. the scalar component of the $N = 1$ chiral superfield contained in the $N = 2$ chiral superfield.

The meromorphic differential λ is [3][4][1](the prime denotes differentiation w.r.t. x)

$$\lambda = \frac{1}{2\pi i}(WF' - 2FW')\frac{(x + b)\, dx}{Fy} \tag{5}$$

where the normalization is chosen such that $(i = 1, \ldots, N_C - 1)$ $a_i = \int_{\alpha_i} \lambda$, $a_{D_i} = \int_{\beta_i} \lambda$ and $\partial_{s_k} a_i = \int_{\alpha_i} \omega_k$, $\partial_{s_k} a_{D_i} = \int_{\beta_i} \omega_k$. $\omega_k = \frac{\partial}{\partial s_k}\lambda = \frac{1}{\pi i}\frac{x^{N_C - k} dx}{y}$, $k = 2, \ldots, N_C$, are a basis of

[1] Here and below relations between abelian differentials are always up to exact differentials

holomorphic differentials (abelian differentials of the first kind) on Σ_{N_C-1}. The constant $b = b(\Lambda, m)$ must be chosen such that for the massless case there are no poles at zeroes of F and the pole at infinity has zero residue. In the massive case λ must have poles at the zeroes of F with residues m_j. One finds that in the massless case $b = 0$. λ also has a double pole at infinity with residue $-\sum m_j$ which vanishes in the massless case. It is, therefore, an abelian differential of the second and third kind in the massless and massive cases, respectively.

The effective (field dependent, dimensionless) gauge coupling is given by the matrix $\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$. $\mathcal{F}$ is thus a homogeneous function of weight two of a_i, m_j, Λ and satisfies the Euler equation[2]

$$2\mathcal{F} = (\Lambda\partial_\Lambda + \sum_j m_j \partial_{m_j} + \sum_i a_i \partial_{a_i})\mathcal{F} \tag{6}$$

Taking derivatives w.r.t. to s_k and using the definition of the a_{D_i} one obtains

$$\frac{\partial}{\partial s_k}(\Lambda\partial_\Lambda + \sum_j m_j \partial_{m_j})\mathcal{F} = \sum_i (a_i \frac{\partial}{\partial s_k} a_{D_i} - a_{D_i} \frac{\partial}{\partial s_k} a_i) \tag{7}$$

Using now the above results we arrive at

$$\frac{\partial}{\partial s_k}(\Lambda\partial_\Lambda + \sum_j m_j \partial_{m_j})\mathcal{F} = \sum_i \int_{\alpha_i} \lambda \int_{\beta_i} \omega_k - \int_{\beta_i} \lambda \int_{\alpha_i} \omega_k \tag{8}$$

The right hand side of this equation can be evaluated with the help of a Riemann bilinear relation [12]. Since they make a distinction between λ being abelian of second or third kind, we will treat the massless and massive cases separately. We first discuss the massless case, where the integrals on the right hand side of eq.(8) can be done explicitly. The mass dependent terms on the left hand side of eqs.(7) and (8) are now absent and λ has a double pole at $w = 1/x = 0$ with expansion

$$\lambda = (\lambda_{-2} w^{-2} + \lambda_0 + \lambda_1 w + \ldots)dw \tag{9}$$

with $\lambda_{-2} = \frac{1}{2\pi i}(2N_C - N_f)$; there are no further poles of λ. The Riemann bilinear relation now reads

$$\sum_i \int_{\alpha_i} \lambda \int_{\beta_i} \omega_k - \int_{\beta_i} \lambda \int_{\alpha_i} \omega_k = 2\pi i \sum_{n\geq 2} \frac{\lambda_{-n}\omega^{(k)}_{n-2}}{n-1} \tag{10}$$

[2] Here and below, Λ is always meant to be Λ_{N_f}.

where $\omega_n^{(k)}$ are the coefficients of ω_k in its expansion around infinity:

$$\omega_k = (\omega_0^{(k)} + \omega_1^{(k)} w + \ldots)dw = \big(-\frac{1}{i\pi} w^{k-2} + O(w^{k-1})\big) \tag{11}$$

i.e. $\omega_0^{(k)} = -\frac{1}{\pi i}\delta_{k,2}$. We then have

$$\partial_{s_k}(\Lambda\partial_\Lambda \mathcal{F}) = 2\pi i \lambda_{-2}\, \omega_0^{(k)} = \frac{i}{\pi}(2N_C - N_f)\delta_{k,2} \tag{12}$$

Integration gives

$$\Lambda\partial_\Lambda \mathcal{F} = \frac{i}{\pi}(2N_C - N_f)s_2 \tag{13}$$

where comparison with the weak coupling expression shows that a possible contribution $const.\,\Lambda^2$ is absent from the right hand side. Let us briefly comment on this result. Taking derivatives with respect to a_i and a_j and using the definition $\partial_{a_i}\partial_{a_j}\mathcal{F} = \tau_{ij} = \frac{1}{2\pi}\theta_{ij} + 4\pi i(\frac{1}{g^2})_{ij}$ one obtains

$$\Lambda\frac{d}{d\Lambda}\tau_{ij} = \frac{i}{2\pi}(2N_C - N_f)\partial_{a_i}\partial_{a_j}\mathrm{Tr}\langle\phi^2\rangle \simeq \frac{i}{\pi}(2N_C - N_f)\mathrm{Tr}(H_i H_j) \tag{14}$$

where in the last step we have taken the semi-classical limit, i.e. have suppressed instanton corrections.

We note that the relation (14) is compatible with perturbation theory. It is well known [13] that $\mathcal{F}$ (or, equivalently, the Wilsonian field dependent gauge coupling) acquires a contribution only at one loop level. This means that s_2 is equal (up to nonperturbative contributions) to its classical value. This agrees with the general observation that correlators of lowest components of gauge invariant chiral superfields are ‘topological’, i.e. they do not depend on positions [14]. Thus they get contributions only from disconnected diagrams. Moreover, they depend holomorphically on the parameters, notably on the gauge coupling. This in fact implies (since there is no dependence on θ in perturbation theory) that there are no perturbative quantum corrections to the classical result. Note, however, that the exact beta function is proportional to $\partial_{a_i}\partial_{a_j}\langle\mathrm{Tr}\phi^2\rangle$, which includes instanton corrections. The above discussion also applies to all the other invariants s_k, and the absence of logarithms, which would have appeared in perturbative contributions, is necessary for them to be globally defined coordinates on the quantum moduli space.

Let us now turn to the remaining classical groups with N_f hypermultiplets in the defining representation $\underline{N}_C$[7]. Here the Riemann surfaces are given by curves of the form [7]

$$xy^2 = W^2 + F \tag{15}$$

where now for $x \to \infty$, $W \sim x^r$ and $F \sim x^{N_f+\nu}$ where $\nu = 4, 3, 0$ for $SO(2r)$, $SO(2r+1)$ and $Sp(2r)$, respectively. The meromorphic differential λ is

$$\lambda = \frac{1}{2\pi i} \frac{WF' - 2W'F}{yF} \, dx \tag{16}$$

with the asymptotic behavior at infinity $\lambda \sim \frac{1}{2\pi i}(l(\text{adj.}) - N_f l(\underline{N}_C)) \frac{dx}{\sqrt{x}}$ where $l(\text{adj.}) = 2(N_C - 2)$, $N_C + 2$ and $l(\underline{N}_C) = 2$, 1 for $SO(N_C)$ and $Sp(N_C)$, respectively. The combination of the indices of the representations appearing in the asymptotic expression of λ, is exactly the one-loop coefficient b_1 of the beta-function for an $N = 2$ supersymmetric gauge theory with N_f hypermultiplets in the defining representation. Introducing the local uniformization variable $x = 1/\xi^2$ one finds that (we are again only considering the massless case here)

$$\lambda_{-2} = -\frac{1}{\pi i}\big(l(\text{adj.}) - N_f l(\underline{N}_C)\big) \, . \tag{17}$$

Likewise one finds the asymptotic behavior of $\omega_k = \partial_{s_k}\lambda$ as $\omega_k = (\omega_0^{(k)} + \omega_1^{(k)}\xi + \ldots)d\xi$ with $\omega_0^{(k)} = -\frac{1}{\pi i}\delta_{k,1}$. Note that in the notation of ref.[7] s_1 is the quadratic invariant: $s_1 = \frac{1}{2}\text{Tr}\langle\phi^2\rangle$. Inserting this into the Riemann relation (8) we get

$$\frac{\partial}{\partial s_k}(\Lambda \partial_\Lambda \mathcal{F}) = \frac{2i}{\pi}\big(l(\text{adj.}) - N_f l(\underline{N}_C)\delta_{k,1}\big) \tag{18}$$

Let us now turn to the massive case. Here we have to use the Riemann bilinear relation for one abelian differential of the first kind (ω_k) and the other of the third kind (λ) with first and second order poles. We will concentrate on the case of $SU(N_C)$. The other groups can be treated similarly. In fact, the meromorphic differential λ now has simple poles at $x_i = m_i$ with residues m_i and a double pole at infinity where it behaves as

$$\lambda = (\lambda_{-2}w^{-2} + \lambda_{-1}w^{-1} + \lambda_0 + \ldots)dw \tag{19}$$

with

$$\lambda_{-2} = \frac{1}{2\pi i}(2N_C - N_f) \qquad \text{and} \qquad \lambda_{-1} = -\frac{1}{2\pi i}\sum_{i=1}^{N_f} m_i$$

The relevant bilinear relation gets contributions from both of these coefficients as well as from the residues of the poles at $x_i = m_i$. The contribution from λ_{-2} is the same as in the massless case. The contribution from the poles at m_i and the pole at infinity is

$$2\pi i \sum_i \text{res}_{x_i}\lambda \int_{x_0}^{x_i} \omega_k + 2\pi i \, \text{res}_\infty \lambda \int_{x_0}^{\infty} \omega_k = -\sum_{i=1}^{N_f} m_i \int_{m_i}^{\infty} \omega_k \tag{20}$$

where x_0 is an arbitrarily chosen point on the Riemann surface [3]. This leads to

$$\frac{\partial}{\partial s_k}\left(\Lambda\partial_\Lambda + \sum_i m_i\partial_{m_i}\right)\mathcal{F} = \frac{i}{\pi}(2N_C - N_f)\delta_{k,2} - \sum_i m_i \int_{m_i}^{\infty}\omega_k \tag{21}$$

Recall that $\omega_k = \partial_{s_k}\lambda$ so that this relation can be integrated w.r.t. s_k leading to a generalization of eq.(13):

$$(\Lambda\partial_\Lambda + \sum_i m_i\partial_{m_i})\mathcal{F} = \frac{i}{2\pi}(2N_C - N_f)\langle \mathrm{Tr}\phi^2\rangle - \sum_i m_i \int_{m_i}^{\infty}\lambda\,. \tag{22}$$

Note that now, in contrast to the massless case, the right hand side seems to depend on all the moduli s_k. We have not attempted to do the remaining integrals explicitly. But let us demonstrate that this expression has in fact the correct decoupling limit. We decouple one of the hypermultiplets by taking the limits, say, $m_{N_f} \equiv M \to \infty$, $\Lambda_{N_f} \to 0$ while keeping $\Lambda_{N_f-1}^{N_C-N_f+1} = M\Lambda_{N_f}^{N_C-N_f}$ fixed. To perform the integral $-M\int_M^\infty \omega_k$ we first change variables $x = M\tilde{x}$ and then perform the decoupling limit. In this limit $y(x) \to M^{N_C}x^{N_C}$ and the integral becomes $\frac{i}{\pi}M^{2-k}\int_1^\infty \frac{d\tilde{x}}{\tilde{x}^k} \to \frac{i}{\pi}\delta_{k,2}$. The integrals for $i = 1,\ldots,N_f-1$ only change in such a way that ω_k turns into the holomorphic differential appropriate for the curve with $N_f - 1$ flavors. We thus find that on the right hand side of eq.(21) we get the change $(2N_C - N_f) \to (2N_C - (N_f - 1))$. The left hand side changes as $\Lambda_{N_f}\partial_{\Lambda_{N_f}} + \sum_{i=1}^{N_f} m_i\partial_{m_i} \to \Lambda_{N_f-1}\partial_{\Lambda_{N_f-1}} + \sum_{i=1}^{N_f-1} m_i\partial_{m_i}$.

Let us now briefly mention that in all cases where $\Lambda\partial_\Lambda\mathcal{F}$ is proportional to u, u is in fact invariant under $Sp(2r;\mathbf{Z})$ transformations $\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} \tilde{a}_d \\ \tilde{a} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}$. This is essentially proven in [8] for $SU(2)$. A simplified version of his proof can be easily generalized to arbitrary groups. From $\tilde{\mathcal{F}}(\tilde{a}) = \tilde{\mathcal{F}}(\tilde{a}(a))$ it follows that $\partial_{a^j}\tilde{\mathcal{F}}(\tilde{a}(a)) = \left(\frac{\partial \tilde{a}^i}{\partial a^j}\right)\tilde{a}_{D_i}$. This relation can be integrated to yield

$$\tilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) + \frac{1}{2}a^T B^T D a + \frac{1}{2}a_D^T C^T A a_D + a^T B^T C a_D \tag{23}$$

This implies that $\mathcal{F} - \frac{1}{2}a^T a_D = \frac{1}{2}\Lambda\partial_\Lambda\mathcal{F}$ is invariant.

Finally, one may consider $N = 2$ models in their Coulomb phase also for matter superfields in representations other than the adjoint or fundamental representations. For those cases it is plausible that the b_1 factor in eq.(22) will be replaced by $2N_c - \sum_i l_i(\mathrm{matter})$.

[3] The independence of the choice follows from the fact that the residues of meromorphic differentials on Riemann surfaces sum up to zero. For more details on this relation, see refs.[12].

The gauge kinetic terms of the low energy effective action of supersymmetric gauge theories in their Coulomb phase can be determined from hyperelliptic curves not only for $N = 2$ supersymmetric models but also for $N = 1$ ones[10]. As in the $N = 2$ case, the ground state of these $N = 1$ models is described by an hyperelliptic quantum moduli space characterized by its singularities and monodromies. The determination of the curve follows from the classical singularities, instantons corrections and the global symmetries of the theory. For instance the curves which correspond to $SU(N_C)$ $N = 1$ models with one adjoint representation and N_f fundamentals (denoted by $(N_{ad} = 1, N_f)$) takes the form of eq.(3)[4]. The corresponding polynomial F is given now by $F = F(x, \Lambda, m_{ij}, Y_{ij})$ where m_{ij} and Y_{ij} are the quark mass matrix and the matrix of Yukawa couplings. When Y is a unit matrix and m is diagonal the model admits an additional supesymmetry. The curves in that case turn into those of $N = 2$ models with N_f hypermultiplets.
Starting with a curve that corresponds to a given $N = 1$ model in its Coulomb phase one can follow the same steps taken above and prove an analogous relation to the one given in eq. (22). We now discuss the relation for certain $N = 1$ classes of models. Using the curves of [4], it turns out that for the class of models $(1, N_f)$ the relation is the same as that given in eq.(22)apart from a replacement of m_i by the eigenvalues of the matrix $Y^{-1}m$.
In case of $(2, 0)$ $N = 1$ models the condition for a Coulomb phase is that the determinant of the adjoint mass matrix vanishes[10]. The curve for $SU(N_C = 2)$[10]is identical to that of the $N = 2$ case with $N_f = 0$ when one replaces $\Lambda^2_{N=2}$ with $\frac{1}{2}\Lambda_{N=1} m_{ad}$. where m_{ad} is the mass of the massive adjoint superfield. A similar situation occurs in the $(2, 1)$ model[11]. We therefore anticipate that the $(2, N_f)$ curves will coincide with those of the $(1, N_f)$ models by a substitution of $\Lambda^{2N_c - N_f}_{N_{ad}=1} \sim \tilde{\Lambda}^{N_c - N_f}_{N_{ad}=2} m^{N_c}_{ad}$
Naively, it seems that the l.h.s of the relation, for instance for $N_f = 0$, takes the form of $\tilde{\Lambda}\partial_{\tilde{\Lambda}}\mathcal{F} + m_{ad}\partial_{m_{ad}}\mathcal{F}$, and on the r.h.s the term proportional to s_2 involves the b_1 pertaining to the one adjoint case. This is quite surprising since apriori we expect such a b_1 to appear only when the massive adjoint decouples. The full determination of the relation and the decoupling for this class of models as well as those which involve other representations is under current investigation.

The relation discussed in this paper appears as a simple partial differential equation for the prepotential $\mathcal{F}$. In order to determine $\mathcal{F}$ completely one needs more equations. Already in the pure $SU(2)$ case one needs one more independent relation. It would be great if one could obtain enough relations which would, in turn, determine $\mathcal{F}$ in a simple way.

Finally we note that while the local counterpart of this relation seems to be quite important [9], the full physical meaning of the relation still alludes us. For fixed Λ, in the massless case, we can rewrite it as

$$(\sum_i a_i \partial_{a_i} - 2)\mathcal{F} = \frac{1}{2\pi i} b_1 \langle \mathrm{Tr}\phi^2 \rangle \tag{24}$$

This equation looks completely quantum mechanical. Moreover, as discussed in this letter, its non-trivial content is associated with the non-perturbative contributions on both sides. The left hand side of (24) looks as if it is related to the "anomalous dimension" of $\mathcal{F}$, i.e. to the deviation of $\mathcal{F}$ from its classical dimension 2. This is due to quantum effects associated with the appearance of Λ. The right hand side involves the beta function. It is tempting to think that one could understand this relation in terms of RG ideas. So far we have not been successful in doing it.

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